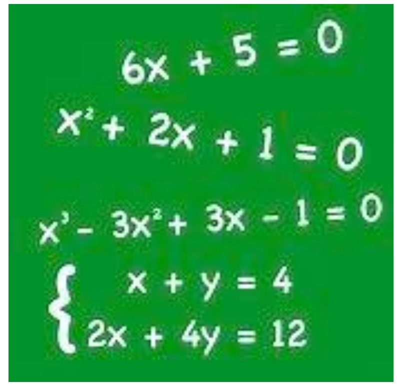
CUBIC AND QUARTIC EQUATIONS



This chapter is a collection of additional equations: Cubic (third degree) and Quartic (fourth degree). Although cubics and quartics can be incredibly difficult to solve, the equations in this chapter can all be solved using the Factoring method. (You can never be too good at factoring!)

❑ *CUBIC EQUATIONS USING THE GCF*

<u>EXAMPLE 1</u>: **Solve the cubic equation:** $30x^3 + 99x^2 = 21x$

<u>Solution</u>: Since factoring is how we solved quadratic equations earlier in the course, let's try factoring here. The factoring method requires one side of the equation to be 0, so we start by subtracting 21x from each side of the equation:

$$30x^3 + 99x^2 - 21x = 0$$

Another theme in this chapter is that factoring always begins with an attempt to factor out the GCF, which in this case is $3x$:

$$3x(10x^2 + 33x - 7) = 0$$ [Check by distributing.]

Factoring the trinomial in the parentheses gives the final factorization of the left side of our equation:

$$3x(5x - 1)(2x + 7) = 0$$

Interpreting $3x$ is one of the factors, we have three factors whose product is 0; we therefore set each of the three factors to 0:

$$3x = 0 \quad \underline{\text{or}} \quad 5x - 1 = 0 \quad \underline{\text{or}} \quad 2x + 7 = 0$$

Solving each of these three linear equations gives us three solutions to our cubic equation:

$$\boxed{x = 0, \frac{1}{5}, -\frac{7}{2}}$$

Homework

1. Solve each cubic equation:

a. $x^3 + 3x^2 + 2x = 0$

b. $4n^3 - 18n^2 + 8n = 0$

c. $x^3 = 16x$

d. $3y^3 = -30y^2 - 75y$

e. $a^3 + 9a = 0$

f. $30x^3 + 25x^2 - 30x = 0$

2. Solve for x: $x^2(x + 1)(2x - 3)(x + 7)^3(x^2 - 4)(x^2 - 5x + 6) = 0$

❑ *Cubic Equations Using Grouping*

EXAMPLE 2: **Solve the cubic equation: $x^3 + 5x^2 = 9x + 45$**

Solution: The first step is to bring the terms on the right side of the equation to the left:

$$x^3 + 5x^2 - 9x - 45 = 0$$

By grouping the first two terms, we can factor the GCF in the first pair of terms and the last pair of terms:

$$x^2(x + 5) - 9(x + 5) = 0$$

Pull out the GCF, $x + 5$:

$$(x + 5)(x^2 - 9) = 0 \qquad \text{(partially factored)}$$

Continue by factoring the difference of squares:

$$(x + 5)(x + 3)(x - 3) = 0 \qquad \text{(fully factored)}$$

Set each factor to 0:

$$x + 5 = 0 \quad \text{or} \quad x + 3 = 0 \quad \text{or} \quad x - 3 = 0$$

Solving each linear equation gives:

$$x = -5 \quad \underline{\text{or}} \quad x = -3 \quad \underline{\text{or}} \quad x = 3$$

And now we have our three solutions:

$$\boxed{x = -5, -3, 3}$$

❑ *QUARTIC EQUATIONS*

<u>EXAMPLE 3:</u> **Solve the quartic equation: $x^4 - 26x^2 + 25 = 0$**

<u>Solution:</u> The factoring we learned in the chapter *Advanced Factoring* is exactly what we need here:

$$x^4 - 26x^2 + 25 = 0$$

$$\Rightarrow \quad (x^2 - 1)(x^2 - 25) = 0$$

Note: Each factor is a difference of squares.

$$\Rightarrow \quad (x + 1)(x - 1)(x + 5)(x - 5) = 0$$

$$\Rightarrow \quad x + 1 = 0 \quad \underline{\text{or}} \quad x - 1 = 0 \quad \underline{\text{or}} \quad x + 5 = 0 \quad \underline{\text{or}} \quad x - 5 = 0$$

$$\Rightarrow \quad x = -1 \quad \text{or} \quad x = 1 \quad \text{or} \quad x = -5 \quad \text{or} \quad x = 5$$

$$\boxed{x = \pm 1, \pm 5}$$

By the way, a fourth-degree polynomial equation can have <u>at most</u> four solutions, but it may have fewer (see the next two examples).

EXAMPLE 4: **Solve for n: $2n^4 - 15n^2 = 27$**

Solution: The factoring technique requires one side of the equation to be 0, so our first step is to make that happen by subtracting 27 from each side of the equation:

$$2n^4 - 15n^2 - 27 = 0$$

$$\Rightarrow \quad (2n^2 + 3)(n^2 - 9) = 0$$

$$\Rightarrow \quad (2n^2 + 3)(n + 3)(n - 3) = 0$$

$$\Rightarrow \quad 2n^2 + 3 = 0 \ \underline{\text{or}} \ n + 3 = 0 \ \underline{\text{or}} \ n - 3 = 0$$

Let's solve the first equation:

$$2n^2 + 3 = 0 \ \Rightarrow \ 2n^2 = -3 \ \Rightarrow \ n^2 = -\frac{3}{2} \ \Rightarrow \ n = \pm\sqrt{-\frac{3}{2}}$$

which are numbers that are NOT in the set of real numbers, ℝ. So, assuming your algebra class involves only real numbers, we conclude that the equation $2n^2 + 3 = 0$ has No Solution. The other two equations should be easy for you by now. The final solution is

$$\boxed{x = \pm 3}$$

EXAMPLE 5: **Solve for a: $a^4 = -13a^2 - 36$**

Solution:

$$a^4 = -13a^2 - 36$$

$$\Rightarrow \quad a^4 + 13a^2 + 36 = 0$$

$$\Rightarrow \quad (a^2 + 9)(a^2 + 4) = 0$$

$$\Rightarrow \quad a^2 + 9 = 0 \text{ or } a^2 + 4 = 0$$

I hope it's clear that neither of these last two equations has a solution in the real numbers, $\mathbb{R}$. We're done:

No Solution

Homework

3. Solve each equation:

a. $x^3 + x^2 - 16x = 16$

b. $n^4 = 13n^2 - 36$

c. $2a^4 - 49a^2 = 25$

d. $x^4 + 17x^2 + 16 = 0$

Review Problems

Solve and check each equation:

4. $30x^3 = 2x^2 + 4x$

5. $n^3 + 5 = 5n^2 + n$

6. $x^4 + 900 = 109x^2$

7. $a^4 + 3a^2 - 4 = 0$

8. $t^4 + 49 + 50t^2 = 0$

9. $w^3 + 2w^2 - 12w - 9 = 0$ [Hard]

Hint: To factor, divide by $w - 3$.

Solutions

1. a. $x = 0, -1, -2$ b. $n = 0, \frac{1}{2}, 4$ c. $x = 0, 4, -4$

d. $y = 0, -5$ e. $a = 0$ f. $x = 0, \frac{2}{3}, -\frac{3}{2}$

2. There are quite a few solutions; you're on your own.

3. a. $x = \pm 4, -1$ b. $n = \pm 2, \pm 3$ c. $a = \pm 5$ d. No solution

4. $x = 0, -\frac{1}{3}, \frac{2}{5}$ **5.** $n = 5, \pm 1$ **6.** $x = \pm 3, \pm 10$

7. $a = \pm 1$ **8.** No solution

9. After you divide the cubic by $w - 3$, the other factor should be quadratic, but it is NOT factorable. So the only solution is $w = 3$.

Edith Hamilton:

"To be able to be

caught up into the

world of thought –
that is educated."